

# $S$ -wave $\pi - \pi$ scattering lengths

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PACS numbers: 13.75 Lb, 11.55 Fv, 11.40 Ha

## Abstract

An old calculation of the  $s$ -wave  $\pi - \pi$  scattering lengths is updated and supplemented with experimental data on the  $\pi - \pi$   $s$ -wave phase shift. The results

$$ma_0 = .215 \quad \text{and} \quad ma_2 = -.039$$

are in excellent agreement with those obtained from a recent analysis of the results published by the Brookhaven E865 collaboration.

The subject of the  $s$ -wave  $\pi - \pi$  scattering lengths has witnessed two main peaks of both experimental and theoretical activity since they were first calculated by Weinberg [1] in 1966. On the experimental side the Geneva-Saclay experiment [2] carried in the mid 1970's gave for the isoscalar scattering length  $ma_0 = .26 \pm .05$ . Recently a new measurement of  $K_{e4}$  decay and of the  $\pi - \pi$  phase-shift difference  $\delta_0^0 - \delta_1^1$  has been published by the Brookhaven E865 collaboration [3] with statistics improved by more than a factor of 10. Analysis and interpretation of the new data has recently been carried out by Descotes et al. [4] who used solutions of the Roy equations [5] obtained by Ananthanarayan et al. [6]. Their results for the scattering lengths are

$$ma_0 = .228 \pm .012 \quad , \quad ma_2 = -.0382 \pm .0038 \quad (1)$$

On the theoretical side considerable activity was devoted to the subject in the late 1960's [7]. Subsequently the effective low energy theory of QCD, Chiral Perturbation Theory (ChPT) was applied to the problem. In ChPT the scattering amplitude is expanded in powers of the momenta and of the quark masses. ChPT is a non renormalizable theory and subtraction constants (low energy couplings) have to be introduced at each order of the calculation. The elaborate evaluation of the perturbation series to two loops was completed only recently [9], [10]. Four low energy couplings  $l_1, l_2, l_3, l_4$  enter in the calculation the values of  $l_1$  and  $l_2$  were obtained in ref. [10] by solving the Roy equations [5].  $l_3$  and  $l_4$  on the other hand are obtained only indirectly.  $l_4$  is expressible in terms of the scalar radius of the pion.

$$r_s^2 = .61 \pm .04 \text{ fm}^2 \quad (2)$$

obtained from an analysis of the s-wave isoscalar phase shifts [11].  $l_3$  is related to the variation of the pion mass  $m$  from its chiral limit. The result obtained in ref. [10] for the scattering lengths is

$$ma_0 = .220 \pm .005 \quad , \quad ma_2 = -.0444 \pm .0010 \quad (3)$$

It is the purpose of this note to point out that the method of collinear dispersion relations [12] as used in [13] is particularly well adapted to the problem of the  $\pi - \pi$  s-wave scattering lengths. It provides a simple alternative to ChPT and yields

$$ma_0 = .215 \quad , \quad ma_2 = -.039 \quad (4)$$

when supplemented with experimental data on the s-wave isoscalar phase-shifts which are used to estimate the variation of the pion isoscalar form factor between momentum transfers  $s = 0$  and  $s = 4m^2$ .

We start from the expression

$$T_{\mu\nu} = \frac{i}{f_\pi^2 m^4} \int d^4 y e^{ip_a y} (m^2 - p_a^2)(m^2 - p_b^2) \langle \pi^c | T A_\mu^a(y) A_\nu^b(0) | \pi^d \rangle \quad (5)$$

where  $f_\pi = .0924 \text{ GeV}$  is the pion decay constant and  $A_\mu^i = 1/2\bar{q}\lambda^i\gamma_\mu q$  denote the axial-vector currents.

The method of collinear dispersion relations [12] consists of writing a dispersion relation in the collinear variable  $x$  in the rest frame of the target pion where

$$p_c = p_d = p \quad , \quad p_a = p_b = xp = q \quad (6)$$

Current Algebra and the generalised Ward-Takahashi identity give

$$x^2 p_\mu p_\nu T_{\mu\nu} = U(x) - \frac{2m^2}{f_\pi^2} .x.(1-x^2)^2 .(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bd}) + \delta_{ab}\delta_{cd} S \quad (7)$$

with

$$U(x) = \frac{i}{f_\pi^2 m^4} \int d^4 y e^{iqy} (m^2 - q^2)^2 \langle \pi^c | T D_a(y) D_b(0) | \pi^d \rangle \quad (8)$$

$$S = \frac{i}{f_\pi^2} \langle \pi | [Q, D(y)]_{e.t} | \pi \rangle \quad (9)$$

where

$$D = \partial_\mu A_\mu, Q = \int d^3 y A(\vec{y}, 0) \quad (10)$$

Eq. (7) yields

$$\lim_{x \rightarrow 0} U(x) = -\delta_{ab}\delta_{cd} S \quad (11)$$

$$\lim_{x \rightarrow 0} \frac{dU(x)}{dx} = -\frac{2m^2}{f_\pi^2} (\delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd}) \quad (12)$$

it is convenient to perform the isospin decomposition of  $U(x)$  as

$$U(x) = A(x)\delta_{ab}\delta_{cd} + B(x)(\delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd}) + x.C(x).(\delta_{ad}\delta_{bc} + \delta_{ac}\delta_{bd}) \quad (13)$$

with  $A$ ,  $B$  and  $C$  even under crossing:  $A(x) = A(-x)$ , etc... It also follows from the definition of  $U$  that

$$U(x = 1) = T_{th} \quad (14)$$

$T_{th}$  denoting the transition matrix element of the process  $\pi^b + \pi^d \rightarrow \pi^a + \pi^c$  at threshold.

Bose symmetry imposes

$$A(x = 1) = B(x = 1) + C(x = 1) \quad (15)$$

and in the soft pion limit

$$\begin{aligned} A(x = 0) &= -S \\ B(x = 0) &= 0 \\ C(x = 0) &= -\frac{2m^2}{f_\pi^2} \end{aligned} \quad (16)$$

In the complex  $x$ -plane,  $A$ ,  $B$  and  $C$  are analytic functions of  $x$  with cuts on the real axis extending from  $-\infty$  to  $-1$  and from  $1$  to  $\infty$ . By Cauchy's theorem then

$$\begin{aligned} A(x = 1) &= -S + \frac{1}{2\pi i} \int_c \frac{dx}{x(x^2 - 1)} A(x) \\ B(x = 1) &= \frac{1}{2\pi i} \int_c \frac{dx}{x(x^2 - 1)} B(x) \\ C(x = 1) &= \frac{-2m^2}{f_\pi^2} \int_c \frac{dx}{x(x^2 - 1)} C(x) \end{aligned} \quad (17)$$

where  $c$  is the contour consisting of straight lines parallel to the real axis immediately above and below the cuts and a circle of large radius  $R$ . The integrals over the circles in the expression above are determined by the asymptotic behaviour of the functions  $A$ ,  $B$  and  $C$ . On the upper and lower arcs  $x^2$  is large and negative so that the operator product expansion for the time ordered product can be used. This constitutes a good approximation except in the vicinity of the real axis. We have

$$\int dy e^{iqy} T D_a(y) D_b(0) \xrightarrow{x^2 \rightarrow -\infty} C_1 + \frac{O_1}{x^2} + \dots, O_1 \propto \bar{q}q \quad (18)$$

The contribution of the unit operator (perturbative) vanishes because only connected parts of the amplitude enter. the integrals over the circles in Eq. (17) amount to  $\langle \pi^c | O_1 | \pi^d \rangle \sim \frac{m_q^2}{m_\pi^2} S$  and are hence negligible. To a good approximation then we can rewrite Eq. (17) making use of crossing symmetry

$$A(x = 1) = -S + \frac{2}{\pi} \int_1^\infty \frac{dx}{x(x^2 - 1)} Abs A(x) \quad (19a)$$

$$B(x = 1) = \frac{2}{\pi} \int_1^\infty \frac{dx}{x(x^2 - 1)} Abs B(x) \quad (19b)$$

$$C(x = 1) = \frac{-2m^2}{f_\pi^2} + \frac{2}{\pi} \int_1^\infty \frac{dx}{x(x^2 - 1)} Abs C(x) \quad (19c)$$

The isospin amplitudes at threshold are linear combinations of the expressions above

$$T_{0,2}^{th} = U_{0,2}(x=1) \quad (20)$$

where

$$\begin{aligned} U_0(x) &= 5B(x) - c(x) \\ U_2(x) &= 2(B(x) - C(x)) \end{aligned} \quad (21)$$

In addition the threshold behaviour of the amplitudes is

$$\begin{aligned} T_{0,2}^{th} &= 32\pi m a_{0,2} \\ \lim_{x \rightarrow 1} Abs U_{0,2}(x) &= \lim_{x \rightarrow 1} Abs T_{0,2}(x) = 32\pi m^2 a_{0,2}^2 \sqrt{(x^2 - 1)} \end{aligned} \quad (22)$$

The second of the expressions above contributes an integrable threshold singularity to the integrals appearing in Eq.(19). Eqs.(19b) and (19c) together with Eq. (21) yield then

$$32\pi m a_0 = \frac{2m^2}{f_\pi^2} + 32\pi m^2 a_0^2 + \frac{2}{\pi} \int_1^\infty \frac{dx}{x(x^2 - 1)} Abs(U_0(x) - U_0(1)) \quad (23)$$

$$32\pi m a_2 = -\frac{4m^2}{f_\pi^2} + 32\pi m^2 a_2^2 + \frac{2}{\pi} \int_1^\infty \frac{dx}{x(x^2 - 1)} Abs(U_2(x) - U_2(1))$$

Because of the constraint Eq.(15) we also have from Eqs.(19a) and (21)

$$32\pi m a_2 = -2S + 32\pi m^2 a_2^2 + \frac{2}{\pi} \int_1^\infty \frac{dx}{x(x^2 - 1)} Abs(A(x) - A(1)) \quad (24)$$

Using the reduction technique,  $Abs U$  can be decomposed into three parts

$$\begin{aligned} Abs U_1 &= \frac{(2\pi)^4}{2} \sum_n \langle \pi^c | j_a | n \rangle \langle n | j_b | \pi^d \rangle \delta(p + q - p_n) - (a \longleftrightarrow b, q \longleftrightarrow -q) \\ Abs U_2 &= \frac{(2\pi)^4}{2} \sum_m (\langle 0 | j_a | m \rangle \langle m, \pi^c | j_b | \pi^d \rangle \\ &\quad + \langle \pi^c | j_a | \pi^d, m \rangle \langle m | j_b | 0 \rangle) \delta(q - p_m) - (a \longleftrightarrow b, q \longleftrightarrow -q) \\ Abs U_3 &= \frac{(2\pi)^4}{2} \sum_l \langle 0 | j_a | l, \pi^d \rangle \langle l, \pi^c | j_b | 0 \rangle \delta(q - p - p_l) - (a \longleftrightarrow b, q \longleftrightarrow -q) \end{aligned} \quad (25)$$

Where  $j = (\square + m^2)D$  and where only connected parts of the matrix elements enter.

$Abs U_1$  contains the usual singularities in the  $s, \bar{s}$  channels.  $Abs U_2$  contains the mass singularities associated with the vertices  $\langle 0 | j | m \rangle$  and  $Abs U_3$  corresponds to the so-called  $Z$  graphs.

The contribution of the single pion state to  $AbsU_2$  vanishes identically, the contribution of the  $0^-$  continuum, the same that provides the corrections to the Goldberger-Treiman relation, is strongly damped and is not expected to amount to more than a few percent of the total and shall be neglected.

The isoscalar channel ( $\sigma(600)$ ,  $f_0(980)$ ,  $f_0(1370)$ ,  $\dots$ ) is expected to practically saturate the contribution of the  $0^+$   $n$  and  $l$  intermediate states. It is readily seen that these states contribute only to  $B(x)$  an amount which we denote by  $b$ . We thus have from Eqs. (23) and (24)

$$\begin{aligned} 32\pi m a_0 &= \frac{2m^2}{f_\pi^2} + 32\pi m^2 a_0^2 + 5b \\ 32\pi m a_2 &= -\frac{4m^2}{f_\pi^2} + 32\pi m^2 a_2^2 + 2b \\ 32\pi m a_2 &= -2S + 32\pi m^2 a_2^2 \end{aligned} \quad (26)$$

If the contribution  $b$  of the continuum were negligible we would deduce from the last two equations above that

$$S \simeq \frac{2m^2}{f_\pi^2} \quad (27)$$

and corresponding values for the scattering lengths which come out close to the ones obtained by Schwinger and by Balachandran et. al [7].

Recall however

$$S = \langle \pi | \sigma | \pi \rangle, \sigma = \frac{i}{f_\pi^2} [Q, D]_{e.t} \quad (28)$$

and consider

$$M(x) = \frac{i}{f_\pi m^2} \int d^4 y e^{iqy} (m^2 - q^2) \langle 0 | D(y) \sigma(0) | \pi(p) \rangle \quad (29)$$

$$M(x=1) = S \quad (30)$$

where the collinear parametrization  $q = px$  has been used once again. The soft pion limit now reads

$$M(x=0) = -\frac{i}{f_\pi} \langle 0 | [Q, \sigma]_{e.t} | \pi \rangle = \frac{m^2}{f_\pi^2} \quad (31)$$

$M(x)$  is an analytic function in the cut complex  $x$ -plane in particular in the interval  $-1 \leq x \leq 3$ . We thus have to  $O(m^2)$

$$M(x=0) = \frac{1}{2} \cdot (M(x=1) + M(x=-1)) \quad (32)$$

$M(x=-1)$  represents the amplitude  $\langle 0 | \sigma | \pi, \pi \rangle_{th}$ . We have then

$$S = \frac{m^2}{f_\pi^2} \cdot (1 - \delta) \quad (33)$$

with

$$\delta = \frac{M(x = -1) - M(x = 1)}{M(x = -1) + M(x = 1)} = O(m^2) \quad (34)$$

so that

$$b = \frac{m^2}{f_\pi^2} \cdot (1 + \delta) \quad (35)$$

It is clear from the above that Eq.(27) cannot hold and that

$$\begin{aligned} 32.\pi.ma_0 &= \frac{7m^2}{f_\pi^2} + 32.\pi.m^2 a_0^2 + \frac{5m^2}{f_\pi^2}.\delta \\ 32.\pi.ma_2 &= -\frac{2m^2}{f_\pi^2} + 32.\pi.m^2 a_2^2 + \frac{2m^2}{f_\pi^2}.\delta \end{aligned} \quad (36)$$

A remark is here in order: to lowest order in  $m^2$  we recover the results of Weinberg [1]

$$ma_0 = \frac{7m^2}{32.\pi.f_\pi^2}, ma_2 = -\frac{m^2}{16.\pi.f_\pi^2} \quad (37)$$

The corrections to the chiral limit thus arise from two sources : a major one proportional to  $m^2 a_{0,2}^2$  coming from the threshold singularity and a minor one due to the variation  $\delta$  of the pion scalar form factor between momentum transfers  $s = 0$  and  $s = 4m^2$ . The results turn out to be quite insensitive to the exact value of  $\delta$  a reliable estimate of which can be inferred assuming elastic unitarity for the pion scalar form factor and using the well known solutions of the Muskhelishvili-Omnès equations [14] which give

$$\frac{M(x = -1)}{M(x = 1)} = \exp\left(\frac{4m^2}{\pi} \int_{4m^2}^{\infty} \frac{\delta_0(s).ds}{s(s - 4m^2)}\right) - 1 \quad (38)$$

$\delta_0$  is the isoscalar s-wave  $\pi - \pi$  phase shift experimental measurements of which are available up to  $s_1 \simeq .8\text{GeV}$  [2], [15]. Taking  $\delta_0(s \geq s_1) = \delta_0(s_1)$  should yield a good approximation for the the rapidly convergent integral appearing in Eq. (38). This gives

$$\delta = .10 \quad (39)$$

$\delta \simeq \frac{4m^2}{6}.r_s^2$  with  $r_s^2$  given by Eq. (2) results in practically the same value.

We finally obtain for the scattering lengths

$$ma_0 = .215, ma_2 = -.039 \quad (40)$$

The error introduced in the numbers above by neglecting the contribution of the  $0^-$  continuum in Eq. (25) should not amount to more than a few percent. That the error due to the neglect of the contribution of the non-threshold intermediate states is small is supported by the fact that no structure is reported in this channel [16]. This is confirmed by the excellent agreement between Eq. (40) and Eq. (1).

We conclude from the above that the method of collinear dispersion relations, provides a simple and reliable alternative to ChPT in the s-wave  $\pi - \pi$  scattering sector and that new low energy  $\pi - \pi$  scattering data are more than welcome [17].

## References

- [1] S. Weinberg, Phys. Rev. Lett. **17**, 616 (1966).
- [2] L. Rosselet et al., Phys. Rev. **D15**, 574 (1977).
- [3] S. Pislak et.al., Phys. Rev. Lett. **87**, 221801(2001).
- [4] S. Descotes, N. H. Fuchs, L. Girlanda and J. Stern; Eur. Phys. Journal **C24**, 469 (2002).
- [5] Phys. Lett. **B36**, 353 (1971).
- [6] B. Ananthanarayan, G. Colangelo, J. Gasser and H. Leutwyler, Phys. Rept. **353**, 207(2001).
- [7] J. Schwinger, Phys. Lett. **B24**, 473(1967); A. P. Balachandran, M. G. Gundzik and F. Nicodemi, Nucl. Phys. **B6**, 557 (1968) and references therein.
- [8] J. Gasser and H. Leutwyler, Ann. Phys. **158**, 142 (1984).
- [9] J. Bijnens, G. Colangelo, G. Ecker, J. Gasser and M. E. Sainio, Phys. Lett. **B374**, 210(1996); Nucl. Phys. **B508**, 263 (1997); ibid. **B517**, 639(E) (1998).
- [10] G. Colangelo, J. Gasser and H. Leutwyler, Nucl. Phys. **B603**, 125 (2001).
- [11] J. F. Donohue, J. Gasser and H. Leutwyler, Nucl. Phys. **B588**, 377 (2000).
- [12] S. Fubini and G. Furlan, Ann. Phys. **48**, 322 (1968).
- [13] N. F. Nasrallah, Nucl. Phys. **B11**, 240 (1969).
- [14] N. I. Muskhelishvili, Singular Integral Equations (Noordhof, Groningen, 1953; R. Omnes, Nuovo Cimento **8**, 316 (1958).
- [15] S. D. Protopopescu et al., Phys. Rev. **D7**, 1279 (1973); B. Hyams et al., Nucl. Phys. **B64**, 134 (1973); W. Ochs, Ph.D. thesis, Ludwig-Maximilians-Universitat (1973); P. Eastbrooks and A. D. Martin, Nucl. Phys. **B79**, 301 (1974).
- [16] Particule Data Group, K. Hagiwara et al., Phys. Rev. **D66**, 010001 (2002).
- [17] B. Adeva et al., CERN proposal CERN/SPSLC 95-1 (1995); <http://dirac.web.cern.ch/DIRAC/>; R. Batley et al., CERN proposal CERN/SPSC 2000-3, CERN/spsc/P253 add.3 (2000)